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On constant-alpha force-free fields in a torus

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Summary

We describe the numerical construction of constant-alpha force-free fields with vanishing normal components on the boundary of a torus by a boundary integral equation approach.

1. Introduction

The equilibrium configuration of an electrically conducting fluid – for instance a plasma – permeated by a sufficiently strong magnetic field is determined mainly through the magnetic force. Since in such a magnetohydrostatic equilibrium there are no other forces to balance the Lorentz force, the magnetic field B must satisfy the equation

$$[\text{curl } B,\ B] = 0 \tag{1.1}$$

and is called a *force-free field* (Lüst and Schlüter [15]). Equation (1.1) can also be written in the form

$$\text{curl } B = \alpha B \tag{1.2}$$

where the scalar α, in general, is space dependent. Making use of div $B = 0$ it follows that (grad α, $B) = 0$, which means that α must be constant along the field lines of B. When α is constant everywhere, the basic equation (1.2) reduces to a linear equation which of course is more easily accessible than the general nonlinear case. Solutions of the linear equation are denoted as *constant-alpha force-free fields*. The study and investigation of constant-alpha force-free fields is of principle interest since it provides a first insight into the possible structure of magnetohydrostatic equilibrium configurations.

Equation (1.1) is studied in astrophysics, for instance, in connection with the investigation of equilibria in sunspots and the possibility of force-free magnetic fields in interstellar space (see [6] and [18]). Force-free fields also are of considerable interest for toroidal equilibrium configurations in the efforts on plasma confinement and controlled thermonuclear fusion (see [8]). In hydrodynamics, solutions of (1.1) are also called *Beltrami fields* and they describe steady incompressible rotational fluid flows with a global Bernoulli constant independent from the streamlines (see [2] and [19]).

In this paper we will consider constant-alpha force-free fields with vanishing normal components on the boundary of a torus, that is, an axially symmetric doubly connected domain. In Section 2 we will establish a general result on the existence and uniqueness of such fields in general doubly connected domains. Our analysis makes results which were previously obtained in [11] more concise. In Section 3 the solution of this boundary-value problem is transformed into a boundary integral equation which in the case of axial symmetry reduces to an equation extended over the boundary of the cross-section of the torus. Finally, in Section 4, the numerical treatment of this integral equation is discussed. For a reasonable approximation we have to appropriately treat the logarithmic singularities of the kernels. This is achieved by extending methods used by Martensen [16] for axially symmetric boundary-value problems in the limiting potential-theoretic case $\alpha = 0$. It also extends the integral-equation method for two-dimensional constant-alpha force-free fields of [13] to the axially symmetric case. Since our integral equation is based on the fundamental solution to the Helmholtz equation our methods also can be applied to axially symmetric boundary-value problems in time-harmonic acoustic and electromagnetic wave propagation. Numerical examples are included at the end of the paper in form of figures for constant-alpha force-free fields in a torus with cross-section given by circles and ellipses.

2. The boundary-value problem

Let D be a doubly connected domain in $\mathbb{R}^3$. We assume the boundary ∂D to be connected and of class $C^{2,\beta}$. By n we denote the unit normal vector to the boundary ∂D directed into the exterior of D. Having in mind that any doubly connected domain is topologically equivalent to a torus we can choose a surface S in D such that simultaneously the set $D \setminus S$ becomes simply connected and the boundary ∂S forms a basis of the first homology group of the complement $\mathbb{R}^3 \setminus D$. Furthermore, we choose a closed curve C, lying on the boundary ∂D and forming a basis of the first homology group of D.

Harmonic vector fields, that means solutions Z to the system

$$\operatorname{div} Z = 0, \quad \operatorname{curl} Z = 0 \quad \text{in } D \tag{2.1}$$

with vanishing normal components

$$(n, Z) = 0 \quad \text{on } \partial D, \tag{2.2}$$

are called Neumann vector fields (see [17]). By (a, b) and $[a, b]$ we denote the usual scalar and vector product of the vectors a and b in $\mathbb{C}^3$, respectively. For a doubly connected domain D there exists exactly one linearly independent Neumann vector field Z and it can be normalized by prescribing its flux

$$\int_S (n, Z)\, \mathrm{d}s = 1 \tag{2.3}$$

through the surface S with normal vector n. In the sequel, we always denote by Z the Neumann field satisfying (2.3).

Consider the following homogeneous Neumann problem for constant-alpha force-free fields: find a vector field $B \in C^1(D) \cap C(\bar{D})$ satisfying

$$\operatorname{curl} B = \alpha B \quad \text{in } D \tag{2.4}$$

with vanishing normal components

$$(n, B) = 0 \quad \text{on } \partial D. \tag{2.5}$$

It is the aim of this paper to first establish that for each α this boundary-value problem is solvable. In the second part we will describe a boundary integral equation approach to numerically construct solutions in the special case of an axially symmetric domain D.

For the existence proof we will need the following three Lemmata (see also [17] and [21]). In the sequel, we always will denote by β a Hölder exponent with $0 < \beta < 1$.

Lemma 2.1. Let $a \in C^{0,\beta}(\bar{D})$ be a divergence-free (in the sense of the limit integral definition) vector field with vanishing normal components $(n, a) = 0$ on ∂D and vanishing flux $\int_S (n, a)\, ds = 0$ through S. Then there exists a unique solution $v \in C^1(D) \cap C(\bar{D})$ of

$$\operatorname{div} v = 0, \quad \operatorname{curl} v = a \quad \text{in } D \tag{2.6}$$

with vanishing tangential components

$$[n, v] = 0 \quad \text{on } \partial D. \tag{2.7}$$

The solution belongs to $C^{1,\beta}(\bar{D})$ and satisfies an a-priori estimate for the Hölder norms

$$\| v \|_{1,\beta} \leqslant C \| a \|_{0,\beta} \tag{2.8}$$

with some constant C. If $a \in C^{1,\beta}(\bar{D})$ then v belongs to $C^2(D)$.

Note, that by Stokes' theorem and the vector identity $(n, \operatorname{curl} v) = -\operatorname{div}[n, v]$ on ∂D the conditions $\operatorname{div} a = 0$ in D, $(n, a) = 0$ on ∂D and $\int_S (n, a)\, ds = 0$ are necessary for the compatibility of (2.6) and (2.7).

Proof: Uniqueness follows from the fact that any curl-free field satisfying (2.7) is also circulation-free and therefore can be represented in the form $v = \operatorname{grad} \varphi$. Then $\operatorname{div} v = 0$ requires $\Delta\varphi = 0$ in D and from the boundary condition (2.7) we have $\varphi = \text{const}$ on ∂D. Hence, $\varphi = \text{const}$ and $v = 0$ in D.

We will prove existence in two steps. First we construct a solution to the inhomogeneous system (2.6). Define the vector potential

$$A(x) := \frac{1}{4\pi} \int_D \frac{a(y)}{|y - x|} ds(y), \quad x \in \mathbb{R}^3,$$

with density a. Using $\operatorname{div} a = 0$ in D and $(n, a) = 0$ on ∂D and employing Gauss' theorem we derive $\operatorname{div} A = 0$ in D. Therefore, $w := \operatorname{curl} A$ satisfies $\operatorname{div} w = 0$ and $\operatorname{curl} w =$

curl curl $A = -\Delta A + \text{grad div}\, A = \text{a}$ in D. From the regularity properties of volume potentials (see [7], Lemma 4.4 and Theorem 6.37) we observe that w belongs to $C^{1,\beta}(\bar{D})$ and satisfies an inequality of the form (2.8).

In the second step we take care of the homogeneous boundary condition (2.7). For the tangential components of w on the boundary we have $\text{div}[n, w] = -(n, \text{curl}\, w) = -(n, a) = 0$ on ∂D. In addition, by Stokes' theorem, we have vanishing circulations $\int_C (t, w)\, \mathrm{d}s = 0$ and $\int_{\partial S}(t, w)\, \mathrm{d}s = \int_S (n, \text{curl}\, w)\, \mathrm{d}s = \int_S (n, a)\, \mathrm{d}s = 0$. Therefore, the tangential components of w on ∂D can be represented in the form

$$w_{tan} = \text{grad}\, f$$

with some function $f \in C^{2,\beta}(\partial D)$. Let u denote the unique solution to the Dirichlet problem $\Delta u = 0$ in D and $u = f$ on ∂D. Then, incorporating inequality (2.8) for w, we have an a-priori estimate (see [7], Theorems 6.6 and 6.14)

$$\| u \|_{2,\beta} \leqslant C_1 \| f \|_{2,\beta} \leqslant C_2 \| a \|_{0,\beta}$$

with some constants C_1 and C_2. Now, $v := w - \text{grad}\, u$ solves the boundary-value problem (2.6) and (2.7) and satisfies an estimate (2.8). If $a \in C^{1,\beta}(\bar{D})$, then w obtained from the volume potential A is in $C^{2,\beta}(\bar{D})$. Therefore, since u is analytic in D, in this case v belongs to $C^2(D)$.

Lemma 2.2. Let $a \in C^{0,\beta}(\bar{D})$ be a divergence-free vector field. Then there exists a unique solution $v \in C^1(D) \cap C(\bar{D})$ of

$$\text{div}\, v = 0, \quad \text{curl}\, v = a \quad \text{in } D \tag{2.9}$$

with vanishing normal components

$$(n, v) = 0 \quad \text{on } \partial D \tag{2.10}$$

and vanishing flux

$$\int_S (n, v)\, \mathrm{d}s = 0 \tag{2.11}$$

through S. The solution belongs to $C^{1,\beta}(\bar{D})$ and satisfies an a-priori estimate

$$\| v \|_{1,\beta} \leqslant C \| a \|_{0,\beta} \tag{2.12}$$

with some constant C.

The proof is similar to Lemma 2.1 and can be found in [14].

Lemma 2.3. Let $a \in C^{0,\beta}(\bar{D})$ be a divergence-free vector field. Then there exists a unique solution $v \in C^2(D) \cap C(\bar{D})$ with curl $v \in C(\bar{D})$ of

$$\text{curl curl}\, v = a, \quad \text{div}\, v = 0 \quad \text{in } D \tag{2.13}$$

Lemma 2.3. Let $a \in C^{0,\beta}(\bar{D})$ be a divergence-free vector field. Then there exists a unique solution $v \in C^2(D) \cap C(\bar{D})$ with curl $v \in C(\bar{D})$ of

$$\text{curl curl}\, v = a, \quad \text{div}\, v = 0 \quad \text{in } D \tag{2.13}$$

with vanishing tangential components

$$[n, v] = 0 \quad \text{on } \partial D. \tag{2.14}$$

The solution and its curl belong to $C^{1,\beta}(\overline{D})$ and satisfy a-priori estimates

$$\| v \|_{1,\beta} \leqslant C \| a \|_{0,\beta}, \quad \|\text{curl } v \|_{1,\beta} \leqslant C \| a \|_{0,\beta} \tag{2.15}$$

with some constant C.

Proof: Firstly, by Lemma 2.2 there exists a unique vector field w satisfying div $w = 0$, curl $w = a$ in D with vanishing normal components $(n, w) = 0$ on ∂D and vanishing flux through S. Secondly, by Lemma 2.1, we find v as the unique solution of div $v = 0$, curl $v = w$ in D with vanishing tangential components $[n, v] = 0$ on ∂D.

Now we are ready to establish our main existence result on constant-alpha force-free fields. We consider the following inhomogeneous Neumann problem: find a vector field $B \in C^1(D) \cap C(\overline{D})$ satisfying the inhomogeneous constant-alpha force-free field equation

$$\text{curl } B - \alpha B = \alpha A \quad \text{in } D \tag{2.16}$$

with normal components

$$(n, B) = g \quad \text{on } \partial D \tag{2.17}$$

and flux

$$\int_S (n, B) \, \mathrm{d}s = f \tag{2.18}$$

through S. Here, $A \in C^{0,\beta}(\overline{D})$ is a given vector field with div $A \in C^{0,\beta}(\overline{D})$, $g \in C^{0,\beta}(\partial D)$ a given scalar function and f a given real number. By Stokes' theorem, the condition

$$\int_{\partial D} \{(n, A) + g\} \, \mathrm{d}s = 0 \tag{2.19}$$

is necessary for the compatibility of (2.16) and (2.17). Therefore, henceforth, we will assume (2.19) to be fulfilled. Note, that in the homogeneous case $A = 0$ and $g = 0$, by Stokes' theorem, the flux condition (2.18) does not depend on the choice for S.

In order to equivalently transform this Neumann boundary-value problem into an equation of the second kind we need to introduce the appropriate operators. By V we denote the normed space

$$V := \{ a \in C^{0,\beta}(D) : \text{div } a = 0 \text{ in } D \}$$

equipped with the usual Hölder norm. Consider the operator $G : V \to V$ mapping $a \in V$ into the solution to the boundary-value problem (2.13) and (2.14) of Lemma 2.3. Because of the a-priori estimate (2.15) the operator $K: V \to V$, defined by $Ka := \operatorname{curl} Ga$, is compact (see [3], Theorem 2.5). Using curl curl $Ga = a$ we find the differential equation

$$\operatorname{curl} Ka = a \quad \text{in } D \tag{2.20}$$

and from $[n, Ga] = 0$ on ∂D, by using $(n, \operatorname{curl} Ga) = -\operatorname{div}[n, Ga]$ on ∂D, we observe the boundary condition

$$(n, Ka) = 0 \quad \text{on } \partial D. \tag{2.21}$$

Finally, from Stokes' theorem and $[n, Ga] = 0$ on ∂D, we derive that Ka has vanishing flux

$$\int_S (n, Ka)\, \mathrm{d}s = 0 \tag{2.22}$$

through S.

Theorem 2.4. *The Neumann boundary-value problem for constant-alpha force-free fields is equivalent to the operator equation of the second kind*

$$(B + A) - \alpha K(B + A) = A + \operatorname{grad} u + \left(f - \int_S \frac{\partial u}{\partial n} \mathrm{d}s\right) Z \tag{2.23}$$

where u denotes a solution of the Neumann problem for Poisson's equation

$$\Delta u = -\operatorname{div} A \quad \text{in } D \tag{2.24}$$

with normal derivative

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial D. \tag{2.25}$$

Proof. We first observe that (2.19) ensures that the solvability condition for the Neumann problem (2.24) and (2.25) is satisfied. The solution is unique up to an additive constant and its gradient belongs to $C^{0,\beta}(\bar{D})$ (see [7], Theorem 6.31). Let B be a solution to the Neumann boundary-value problem (2.16) to (2.18) and set

$$b := B - \alpha K(B + A) - \operatorname{grad} u - \left(f - \int_S \frac{\partial u}{\partial n} \mathrm{d}s\right) Z.$$

Note that b is well defined since $\operatorname{div}(B + A) = 0$ in D. Then, from (2.1) to (2.3), (2.16) to (2.18), (2.20) to (2.22), and (2.24) to (2.25), we find that b satisfies $\operatorname{div} b = 0$ and $\operatorname{curl} b = 0$ in D and has vanishing normal components $(n, b) = 0$ on ∂D and flux $\int_S (n, b)\, \mathrm{d}s = 0$ through S. Therefore, the Neumann vector field b must be zero, that means B solves equation (2.23).

Conversely, let B solve the equation (2.23). Note, that the right-hand side of (2.23) belongs to V because of (2.24). Then, again using (2.1) to (2.3), (2.20) to (2.22) and (2.25) we find that B satisfies $\operatorname{curl} B - \alpha B = \alpha A$ in D, has normal components $(n, B) = g$ on ∂D and flux $\int_S (n, B)\, ds = f$ through S, that means B solves the boundary-value problem (2.16) to (2.18).

For the further investigation of equation (2.23) we introduce a scalar product $(\cdot, \cdot)_V$ on V by

$$(a, b)_V := \int_D (\operatorname{curl} Ga, \operatorname{curl} Gb)\, dx.$$

Since $(a, a)_V = 0$ implies $\operatorname{curl} Ga = 0$ in D, from $\operatorname{div} Ga = 0$ in D and $[n, Ga] = 0$ on D and Lemma 2.1 it follows that $Ga = 0$ in D. Then we have $a = \operatorname{curl} \operatorname{curl} Ga = 0$ in D, that means, $(\cdot, \cdot)_V$ indeed is positive definite. The operator K is self-adjoint with respect to this scalar product. This follows from Gauss' theorem:

$$\begin{aligned}(Ka, b)_V &= \int_D (\operatorname{curl}(G \operatorname{curl} Ga), \operatorname{curl} Gb)\, dx \\ &= \int_D (\operatorname{curl} \operatorname{curl}(G \operatorname{curl} Ga), Gb)\, dx + \int_{\partial D} (\operatorname{curl}(G \operatorname{curl} Ga), [n, Gb])\, ds \\ &= \int_D (\operatorname{curl} Ga, Gb)\, dx = \int_D (Ga, \operatorname{curl} Gb)\, dx + \int_{\partial D} ([n, Ga], Gb)\, ds \\ &= \int_D (Ga, \operatorname{curl} Gb)\, dx.\end{aligned}$$

Theorem 2.5. *There exists a countable set of real numbers α, accumulating only at infinity, for which the homogeneous constant-alpha force-free field equation*

$$\operatorname{curl} B = \alpha B \quad \textit{in } D \tag{2.26}$$

with homogeneous boundary condition

$$(n, B) = 0 \quad \textit{on } \partial D \tag{2.27}$$

and vanishing flux

$$\int_S (n, B)\, ds = 0 \tag{2.28}$$

through S has a finite number of linearly independent solutions. For all numbers α not belonging to this set of eigenvalues there exists a unique solution of the inhomogeneous problem (2.16) *to* (2.18) *for all inhomogeneities satisfying the compatibility condition* (2.19).

Proof. The existence of eigenvalues follows by standard arguments for compact self adjoint operators in positive dual systems (see [9], Theorem 5.20): there exists a countable set of real numbers α, accumulating only at infinity, for which the homogeneous equation

$$B - \alpha K B = 0$$

has a finite-dimensional null space. For all values of α different from these eigenvalues, by the Riesz theory for compact operators (see [3]) the inhomogeneous equation (2.23) has a unique solution B. This, in view of the equivalence Theorem 2.4, completes our proof.

Remark 2.6. By the Fredholm theory for compact operators (see [3]) in the case where α is an eigenvalue, the inhomogeneous problem is solvable if and only if the condition

$$\int_D \left(GB,\ A + \operatorname{grad} u + \left(f - \int_S \frac{\partial u}{\partial n} \mathrm{d}s \right) Z \right) \mathrm{d}x = 0 \tag{2.29}$$

is satisfied for all solutions B to the homogeneous problem (2.26) to (2.28). Condition (2.29) can be seen to be equivalent to the form given in Theorem 6.2 of [12].

Summarizing, we see that for each α the homogeneous Neumann problem (2.4) and (2.5) for constant-alpha force-free fields has a solution. For those α not belonging to the set of eigenvalues from Theorem 2.5 there exactly one linearly independent solution with flux different from zero. If α is an eigenvalue of multiplicity m then obviously we have m linearly independent solutions of (2.4) and (2.5) with vanishing flux. In addition, there exists a further solution with flux different from zero in the case where the solvability condition (2.29) is satisfied for $A = 0$ and $u = 0$ in D.

Remark 2.7. From our analysis it is clear that solutions to the inhomogeneous Neumann problem for constant-alpha force-free fields are real-valued provided the inhomogeneities are real-valued. For the eigenvalues the solutions to the homogeneous problem also may be chosen to be real-valued.

3. Axially symmetric solutions

For the remainder of this paper we will consider the case where the domain $D \subset \mathbb{R}^3$ is axially symmetric with cross-section $S \subset \mathbb{R}^2$. For those values of α for which the homogeneous problem (2.26) to (2.28) admits only the trivial solution the fields satisfying (2.4) and (2.5) must be axially symmetric because of uniqueness reasons. Note that solutions of (2.26) to (2.28) may be non axially symmetric.

The axially symmetric case can be reduced to solving a two-dimensional Dirichlet boundary-value problem in the cross-section S. By using cylindrical coordinates (r, φ, z), it is easily verified that the homogeneous Neumann constant-alpha force-free field problem

$$\operatorname{curl} B = \alpha B \quad \text{in } D \tag{3.1}$$

and

$$(n, B) = 0 \quad \text{on } \partial D \tag{3.2}$$

is equivalent to solving the Dirichlet problem

$$\frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial}{\partial r}(rB_\varphi) \right) + \frac{\partial^2 B_\varphi}{\partial z^2} + \alpha^2 B_\varphi = 0 \quad \text{in } S \tag{3.3}$$

with boundary condition

$$t_r \frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi) + t_z \frac{\partial B_\varphi}{\partial z} = 0 \quad \text{on } \partial S \tag{3.4}$$

for the component B_φ in φ-direction and then taking

$$B_r = -\frac{1}{\alpha}\frac{\partial B_\varphi}{\partial z} \quad \text{and } B_z = \frac{1}{\alpha r}\frac{\partial}{\partial r}(rB_\varphi)$$

for the components B_r and B_z in r- and z-direction. Here, $t = (t_r, t_z)$ denotes the unit tangential vector to the boundary ∂S such that $[t, n] = e_\varphi$, with e_φ the unit vector in φ-direction. The boundary condition (3.4) is equivalent to

$$rB_\varphi = \text{const} \quad \text{on } \partial S. \tag{3.5}$$

Hence, for each α the homogeneous problem (3.1) and (3.2) has axially symmetric solutions. Note that there exists a countable set of Dirichlet eigenvalues, accumulating only at infinity, for which the equation (3.3) admits nontrivial solutions with homogeneous boundary condition

$$B_\varphi = 0 \quad \text{on } \partial S. \tag{3.6}$$

For those α not belonging to these Dirichlet eigenvalues there exists exactly one linearly independent axially symmetric solution to (3.1) and (3.2) and this solution is characterized by the property $rB_\varphi = \text{const} \neq 0$ on ∂D.

If α is such a Dirichlet eigenvalue with multiplicity m then obviously the homogeneous problem (3.1) and (3.2) has m linearly independent axially symmetric solutions with the property $B_\varphi = 0$ on ∂D. In addition, there exists a further nontrivial solution with $rB_\varphi = \text{const} \neq 0$ on ∂D in the case where the solvability condition for the inhomogeneous Dirichlet problem (3.5), namely

$$\int_{\partial S} \frac{1}{r}\frac{\partial}{\partial n} rB_\varphi \, \mathrm{d}s = 0$$

is satisfied for all solutions B_φ to the homogeneous problem (3.6).

We will now derive a boundary integral equation for these axially symmetric solutions which is uniquely solvable for all α different from a Dirichlet eigenvalue for (3.3). This integral equation will be based on the representation formula

$$B(x) = \operatorname{curl}\int_{\partial D} [B(y), n(y)]\,\Phi(x, y)\, \mathrm{d}s(y)$$

$$+\alpha\int_{\partial D} [B(y), n(y)]\,\Phi(x, y)\, \mathrm{d}s(y), \quad x \in D, \tag{3.7}$$

for solutions $B \in C^1(D) \cap C(\overline{D})$ of the constant-alpha force-free field equation (3.1) with vanishing normal components (3.2) on ∂D. Here,

$$\Phi(x, y) = \frac{1}{4\pi} \frac{e^{i\alpha|x-y|}}{|x-y|}, \quad x \neq y,$$

denotes the fundamental solution to the Helmholtz equation. A proof of (3.7) is contained in [13]. It also can be derived from the Stratton-Chu representation formula for solutions to the vector Helmholtz equation (see [3], Theorem 4.11). Letting x tend to the boundary with the aid of the jump relations for single-layer vector potentials (see [3], Theorem 2.26) we derive the integral equation

$$\gamma(x) + 2\int_{\partial D} [n(x), \operatorname{curl}_x\{\Phi(x, y)\gamma(y)\} + \alpha\Phi(x, y)\gamma(y)]\, ds(y) = 0,$$

$$x \in \partial D, \tag{3.8}$$

for the unknown tangential component

$$\gamma := [B, n]$$

of B on the boundary ∂D. For the axially symmetric case we can write

$$\gamma = \eta t + \delta e_\varphi \quad \text{on } \partial S,$$

that is

$$B_{\tan} = \delta t - \eta e_\varphi \quad \text{on } \partial S.$$

From [3.5] we already know that

$$r\eta = \text{const} \quad \text{on } \partial S. \tag{3.9}$$

From [3.5] we already know that

$$r\eta = \text{const} \quad \text{on } \partial S. \tag{3.9}$$

Therefore, we only need to determine the remaining unknown component δ from the integral equation (3.8). For fixed $x \in \partial D$ the functions $(t(x), e_\varphi(y))$ and $([t(x), t(y)], x - y)$ are odd functions with respect to $y \in \partial D$. Therefore, from symmetry reasons we see that

$$\int_{\partial D} (t(x), e_\varphi(y))\Phi(x, y)\psi(y)\, ds(y) = 0, \quad x \in \partial D, \tag{3.10}$$

and

$$\int_{\partial D} (t(x), [\operatorname{grad}_x\Phi(x, y), t(y)])\psi(y)\, ds(y) = 0, \quad x \in \partial D, \tag{3.11}$$

for axially symmetric functions ψ. Hence, taking the e_φ-component of (3.8), we arrive at the scalar integral equation

$$\delta(x) - 2\int_{\partial D}\big(t(x), [\operatorname{grad}_x \Phi(x, y), e_\varphi(y)]\big)\delta(y)\, ds(y)$$
$$= 2\alpha\int_{\partial D}(t(x), t(y))\Phi(x, y)\eta(y)\, ds(y), \quad x \in \partial D, \tag{3.12}$$

for the unknown δ. Since we have already established existence of solutions to the homogeneous Neumann problem (3.1) and (3.2), given η through (3.9), the existence of solutions to the integral equation (3.12) is immediate. Hence we need only be concerned with the question of uniqueness. Once the integral equation is solved, the solution to the boundary-value problem is obtained through (3.7).

Let δ be an axially symmetric solution to the homogeneous equation

$$\delta(x) - 2\int_{\partial D}\big(t(x), [\operatorname{grad}_x \Phi(x, y), e_\varphi(y)]\big)\delta(y)\, ds(y) = 0, \quad x \in \partial D.$$

Defining $\gamma := \delta e_\varphi$ and again employing the symmetry relation (3.11), we see that γ solves the homogeneous vector integral equation

$$\gamma(x) + 2\int_{\partial D}[n(x), \operatorname{curl}_x\{\Phi(x, y)\gamma(y)\}]\, ds(y) = 0, \quad x \in \partial D.$$

Now by the jump relations, the last equation implies that the field

$$E(x) := \operatorname{curl}\int_{\partial D}\gamma(y)\Phi(x, y)\, ds(y), \quad x \in \mathbb{R}^3\setminus\partial D,$$

has vanishing tangential components

$$[n, E_+] = 0 \quad \text{on } \partial D.$$

Here, by the indices + and − we distinguish the limits obtained when approaching the boundary ∂D from inside $\mathbb{R}^3\setminus\bar{D}$ and D, respectively. Since E satisfies the Silver-Müller radiation condition, by the uniqueness for the exterior boundary-value problem for the reflection of electromagnetic fields at perfect conductors we conclude that $E = 0$ in the exterior domain $\mathbb{R}^3\setminus D$ (see [3], Theorem 4.18). Then by the jump relations we get

$$[n, \operatorname{curl} E_-] = 0 \quad \text{on } \partial D.$$

Hence, $B := \operatorname{curl} E$ is an axially symmetric solution to the Maxwell equations

$$\operatorname{curl}\operatorname{curl} B - \alpha^2 B = 0 \quad \text{in } D \tag{3.13}$$

with vanishing tangential components

$$[n, B] = 0 \quad \text{on } \partial D. \tag{3.14}$$

Since B has only a nonvanishing component B_φ in e_φ-direction, the equation (3.13) coincides with (3.4) and the boundary condition (3.14) requires $B_\varphi = 0$ on ∂D. Therefore, provided α is not a Dirichlet eigenvalue for (3.4) we have $B_\varphi = 0$ in D and from curl curl $E - \alpha^2 E = 0$ we also see that $E = 0$ in D. Now, finally the jump relations yield $\gamma = [n, E_+] - [n, E_-] = 0$ on ∂D and therefore $\delta = 0$ on ∂D. Note, that in the case where α is a Dirichlet eigenvalue we do not have uniqueness for (3.12). We can summarize our results into the following

Theorem 3.1. *The integral equation* (3.12) *with the right-hand side given through* (3.9) *has a unique solution provided* α *is not a Dirichlet eigenvalue for* (3.4). *Its solution gives the tangential component of a constant-alpha force-free field with vanishing normal components on the boundary.*

Note that, due to the axial symmetry, equation (3.12) of course essentially is an integral equation over the boundary ∂S of the cross-section S.

The integral equation (3.12) has complex-valued kernels despite the fact that by Remark 2.7 the solutions are real-valued. The reason for this lies in the fact that for the uniqueness proof we need to incorporate the radiation condition which requires the complex-valued fundamental solution to the Helmholtz equation.

4. Numerical solution of the integral equation

In this final section we will describe the numerical solution of the integral equation (3.12). Let

$$x(\sigma) = (r(\sigma),\ z(\sigma)),\ 0 \leqslant \sigma \leqslant 2\pi,$$

be a parametric representation of the boundary ∂S to the cross-section S. Note that $r(\sigma) > 0$ for all $0 \leqslant \sigma \leqslant 2\pi$., Then, the boundary ∂D of the axially symmetric domain D is given by

$$x(\sigma, \varphi) = (r(\sigma) \cos \varphi,\ r(\sigma) \sin \varphi,\ z(\sigma)),\ 0 \leqslant \sigma,\ \varphi \leqslant 2\pi.$$

It is a matter of straightforward calculations to transform the axially symmetric surface integrals over ∂D into integrals over the parameter domain $[0, 2\pi]$. Introduce

$$R(\sigma, \tau, \varphi) := \left\{[r(\sigma)]^2 - 2r(\sigma)r(\tau) \cos \varphi + [r(\tau)]^2 + [z(\sigma) - z(\tau)]^2\right\}^{1/2}.$$

Then the integral equation (3.12) takes the form

$$\tilde{\delta}(\sigma) - \frac{1}{2\pi}\int_0^{2\pi} H(\sigma, \tau)\tilde{\delta}(\tau)\, \mathrm{d}\tau = \frac{\alpha}{2\pi}\int_0^{2\pi} L(\sigma, \tau)\tilde{\eta}(\tau)\, \mathrm{d}\tau,\ 0 \leqslant \sigma \leqslant 2\pi, \tag{4.1}$$

for $\tilde{\delta}(\sigma) := \{[r'(\sigma)]^2 + [z'(\sigma)]^2\}^{1/2}\delta(x(\sigma))$ and $\tilde{\eta}(\sigma) := \eta(x(\sigma))$. The kernels are given by

$$H(\sigma, \tau) := \int_0^{2\pi} r(\tau)\{z'(\sigma)[r(\sigma) \cos \varphi - r(\tau)] + [z(\tau) - z(\sigma)]r'(\sigma) \cos \varphi\}$$

$$\times \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\frac{\mathrm{e}^{\mathrm{i}\alpha R}}{R}\mathrm{d}\varphi \tag{4.2}$$

and

$$L(\sigma, \tau) := \int_0^{2\pi} r(\tau)\{r'(\sigma)r'(\tau) \cos \varphi + z'(\sigma)z'(\tau)\} \frac{e^{i\alpha R}}{R} d\varphi. \tag{4.3}$$

For a satisfactory numerical approximation a careful investigation of the nature of the singularity of the kernels is necessary. Since essentially we are solving a two-dimensional boundary-value problem in the cross-section S we expect a logarithmic singularity when $\sigma = \tau$. Our analysis extends a procedure used by Martensen [16] in the limiting potential-theoretic case $\alpha = 0$. For integers m we define integrals

$$I_m(\sigma, r) := \int_0^{2\pi} [R(\sigma, \tau, \varphi)]^{m-1} d\varphi$$

and

$$J_m(\sigma, \tau) := \int_0^{2\pi} \cos \varphi [R(\sigma, \tau, \varphi)]^{m-1} d\varphi.$$

Then, through partial integration, we readily derive the recurrence relations

$$I_{m+2} = pI_m - qJ_m \tag{4.4}$$

and

$$(m+3)J_{m+2} = (m+1)[pJ_m - qI_m] \tag{4.5}$$

for all m. Here, we have set

$$p(\sigma, \tau) := [r(\sigma)]^2 + [r(\tau)]^2 + [z(\sigma) - z(\tau)]^2$$

and

$$q(\sigma, \tau) := 2r(\sigma)r(\tau).$$

For even indices the initial terms are given through

$$I_0 = \frac{4}{(p+q)^{1/2}} K(k)$$

and

$$J_0 = \frac{4}{q(p+q)^{1/2}} [pK(k) - (p+q)E(k)]$$

where

$$k(\sigma, \tau) := \left(\frac{2q(\sigma, \tau)}{p(\sigma, \tau) + q(\sigma, \tau)} \right)^{1/2}$$

and where E and K denote the complete elliptic integrals

$$E(k)=\int_0^{\pi/2}(1-k^2\sin^2\varphi)^{1/2}\,\mathrm{d}\varphi$$

and

$$K(k)=\int_0^{\pi/2}(1-k^2\sin^2\varphi)^{-1/2}\,\mathrm{d}\varphi$$

with modulus $0\leqslant k<1$. Hence, for m even we can split

$$I_m=I_m^E E(k)+I_m^K K(k),\quad J_m=J_m^E E(k)+J_m^K K(k)$$

where the coefficients satisfy the recurrence relations (4.4) and (4.5) with initial terms

$$I_0^E=0,\quad I_0^K=\frac{4}{(p+q)^{1/2}},$$

and

$$J_0^E=-\frac{4}{q}(p+q)^{1/2},\quad J_0^K=\frac{4p}{q(p+q)^{1/2}}.$$

For odd indices we have the initial terms $I_1=2\pi$ and $J_1=0$.

Integrating the power series for the exponential term by term we find

$$L(\sigma,\tau)=L^E(\sigma,\tau)E(k)+L^K(\sigma,\tau)K(k)+L^i(\sigma,\tau) \tag{4.6}$$

where

$$L^E(\sigma,\tau)=r(\tau)\Bigg[z'(\sigma)z'(\tau)\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^{2m}}{(2m)!}I_{2m}^E + r'(\sigma)r'(\tau)\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^{2m}}{(2m)!}J_{2m}^E\Bigg],$$

$$L^K(\sigma,\tau)=r(\tau)\Bigg[z'(\sigma)z'(\tau)\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^{2m}}{(2m)!}I_{2m}^K + r'(\sigma)r'(\tau)\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^{2m}}{(2m)!}J_{2m}^K,\Bigg],$$

$$L^i(\sigma,\tau)=\mathrm{i}r(\tau)\Bigg[z'(\sigma)z'(\tau)\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^{2m+1}}{(2m+1)!}I_{2m+1} + r'(\sigma)r'(\tau)\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^{2m+1}}{(2m+1)!}J_{2m+1}\Bigg].$$

These series can be numerically evaluated with the help of the recurrence relations for I_m and J_m. Note that (4.4) and (4.5) are unstable, but they can be transformed into recurrence relations for

$$\tilde{I}_m := \frac{\alpha^m}{m!} I_m \quad \text{and } \tilde{J}_m := \frac{\alpha^m}{m!} J_m$$

which turn out to be stable for α not too large.

Similarly we derive

$$H(\sigma, \tau) = H^E(\sigma, \tau) E(k) + H^K(\sigma, \tau) K(k) + H^i(\sigma, \tau) \tag{4.7}$$

where

$$\begin{aligned} H^E(\sigma, \tau) = M(\sigma, \tau) + r(\tau) \Bigg[& r(\tau) z'(\sigma) \alpha^2 \sum_{m=0}^{\infty} \frac{(-1)^m \alpha^{2m}}{(2m)!} \frac{I_{2m}^E}{2m+2} \\ & \times \{ [z(\sigma) - z(\tau)] r'(\sigma) - r(\sigma) z'(\sigma) \} \\ & \times \left\{ \frac{4}{q(p+q)^{1/2}} + \alpha^2 \sum_{m=0}^{\infty} \frac{(-1)^m \alpha^{2m}}{(2m)!} \frac{J_{2m}^E}{2m+2} \right\} \Bigg], \end{aligned}$$

$$\begin{aligned} H^K(\sigma, \tau) = r(\tau) \Bigg[& r(\tau) z'(\sigma) \alpha^2 \sum_{m=0}^{\infty} \frac{(-1)^m \alpha^{2m}}{(2m)!} \frac{I_{2m}^K}{2m+2} \\ & \times \{ [z(\sigma) - z(\tau)] r'(\sigma) - r(\sigma) z'(\sigma) \} \\ & \times \left\{ \frac{-4}{q(p+q)^{1/2}} + \alpha^2 \sum_{m=0}^{\infty} \frac{(-1)^m \alpha^{2m}}{(2m)!} \frac{J_{2m}^K}{2m+2} \right\} \Bigg], \end{aligned}$$

$$\begin{aligned} H^i(\sigma, \tau) = \mathrm{i}\alpha^2 r(\tau) \Bigg[& r(\tau) z'(\sigma) \sum_{m0}^{\infty} \frac{(-1)^m \alpha^{2m+1}}{(2m+1)!} \frac{I_{2m+1}}{2m+3} \\ & \times \{ [z(\sigma) - z(\tau)] r'(\sigma) - r(\sigma) z'(\sigma) \} \\ & \times \sum_{m=0}^{\infty} \frac{(-1)^m \alpha^{2m+1}}{(2m+1)!} \frac{J_{2m+1}}{2m+3} \Bigg]. \end{aligned}$$

For the leading term M in the expression for H^E from

$$I_{-2} = \frac{4}{(p+q)^{1/2}} \frac{1}{p-q} E(k)$$

and

$$J_{-2} = \frac{4}{(p+q)^{1/2}}\left[\frac{1}{p-q}E(k) - \frac{1}{q}\big(K(k) - E(k)\big)\right]$$

we obtain

$$M(\sigma,\tau) = \frac{4r(\tau)\{[r(\tau)-r(\sigma)]z'(\sigma) + [z(\sigma)-z(\tau)]r'(\sigma)\}}{[p(\sigma,\tau)+q(\sigma,\tau)]^{1/2}\{[r(\sigma)-r(\tau)]^2 + [z(\sigma)-z(\tau)]^2\}}, \quad \sigma \neq \tau.$$

$$M(\sigma,\sigma) = \frac{z'(\sigma)r''(\sigma) - r'(\sigma)z''(\sigma)}{[r'(\sigma)]^2 + [z'(\sigma)]^2}.$$

For $k = 1$, that is for $\sigma = \tau$, the complete elliptic integrals have logarithmic singularities of the form (see [5])

$$E(k) = \frac{2}{\pi}\ln\frac{4}{k'}[K(k') - E(k')] + \tilde{E}(k')$$

and

$$K(k) = \frac{2}{\pi}\ln\frac{4}{k'}K(k') + \tilde{K}(k')$$

where k' denotes the complementary modulus

$$k' = (1-k^2)^{1/2}$$

and where $\tilde{E}$ and $\tilde{K}$ are analytic functions for $0 \leqslant k' < 1$ with $\tilde{E}(0) = 1$ and $\tilde{K}(0) = 0$. In particular, there holds $E(1) = 1$ and $\lim_{k\to 1}[K(k) - \ln 4/k'] = 0$. We can transform

$$2\ln\frac{4}{k'} = -\ln\left(4\sin^2\frac{\sigma-\tau}{2}\right) + g(\sigma,\tau)$$

with a smooth function g given by

$$g(\sigma,\tau) = \ln\left[64\sin^2\frac{\sigma-\tau}{2}\,\frac{p(\sigma,\tau)+q(\sigma,\tau)}{p(\sigma,\tau)-q(\sigma,\tau)}\right], \quad \sigma \neq \tau,$$

$$g(\sigma,\sigma) = \ln\frac{64[r(\sigma)]^2}{[r'(\sigma)]^2 + [z'(\sigma)]^2}.$$

Therefore we finally can split the kernels of (4.1) in the form

$$H(\sigma,\tau) = H_1(\sigma,\tau)\ln\left(4\sin^2\frac{\sigma-\tau}{2}\right) + H_2(\sigma,\tau) \tag{4.8}$$

and

$$L(\sigma, \tau) = L_1(\sigma, \tau) \ln\left(4 \sin^2 \frac{\sigma - \tau}{2}\right) + L_2(\sigma, \tau) \tag{4.9}$$

with

$$H_1(\sigma, \tau) = -\frac{1}{\pi} H^E(\sigma, \tau)[K(k') - E(k')] - \frac{1}{\pi} H^K(\sigma, \tau) K(k'),$$

$$L_1(\sigma, \tau) = -\frac{1}{\pi} L^E(\sigma, \tau)[K(k') - E(k')] - \frac{1}{\pi} L^K(\sigma, \tau) K(k'),$$

$$H_2(\sigma, \tau) = H(\sigma, \tau) - H_1(\sigma, \tau) \ln\left(4 \sin^2 \frac{\sigma - \tau}{2}\right), \quad \sigma \neq \tau,$$

$$H_2(\sigma, \sigma) = H^E(\sigma, \sigma) + \frac{1}{2} \ln \frac{64[r(\sigma)]^2}{[r'(\sigma)]^2 + [z'(\sigma)]^2} H^K(\sigma, \sigma) + H^i(\sigma, \sigma).$$

$$L_2(\sigma, \tau) = L(\sigma, \tau) - L_1(\sigma, \tau) \ln\left(4 \sin^2 \frac{\sigma - \tau}{2}\right), \quad \sigma \neq \tau,$$

$$L_2(\sigma, \sigma) = L^E(\sigma, \sigma) + \frac{1}{2} \ln \frac{64[r(\sigma)]^2}{[r'(\sigma)]^2 + [z'(\sigma)]^2} L^K(\sigma, \sigma) + L^i(\sigma, \sigma).$$

In this decomposition the coefficients H_1, H_2, L_1 and L_2 are continuous functions provided the boundary ∂S is of class C^2. If ∂S is analytic then the coefficients are analytic.

For the numerical approximation we choose an equidistant set of knots

$$\sigma_k := \frac{\pi}{N} k, \quad k = 0, \ldots, 2N - 1,$$

and choose the quadrature formulae

$$\frac{1}{2\pi} \int_0^{2\pi} f(\sigma)\, \mathrm{d}\sigma \approx \frac{1}{2N} \sum_{k=0}^{2N-1} f(\sigma_k) \tag{4.10}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} f(\sigma) \ln\left(4 \sin^2 \frac{\sigma}{2}\right) \mathrm{d}\sigma \approx \sum_{k=0}^{2N-1} R_k f(\sigma_k) \tag{4.11}$$

where the weights R_k are given by

$$R_k = -\frac{1}{N} \left\{ \frac{(-1)^k}{2N} + \sum_{j=1}^{N-1} \frac{\cos j\sigma_k}{j} \right\}, \quad k = 0, \ldots, 2N - 1.$$

These quadrature rules are obtained by replacing f by its trigonometric interpolation polynomial and then integrating analytically (see Martensen [16]). Provided f is analytic, according to derivative-free error estimates for the remainder term in trigonometric interpolation (see Kress [10]) in the spirit of Davis' method [4], the error of the quadrature rules (4.10) and (4.11) decreases at least exponentially when the number $2N$ of knots is increased.

We employ Nyström's quadrature method and replace (4.1) by the approximating linear system

$$\delta_k - \sum_{j=0}^{2N-1} \left\{ R_{|j-k|} H_1(\sigma_k, \sigma_j) + \frac{1}{2N} H_2(\sigma_k, \sigma_j) \right\} \delta_j$$

$$= \alpha \sum_{j=0}^{2N-1} \left\{ R_{|j-k|} L_1(\sigma_k, \sigma_j) + \frac{1}{2N} L_2(\sigma_k, \sigma_j) \right\} \eta_j \tag{4.12}$$

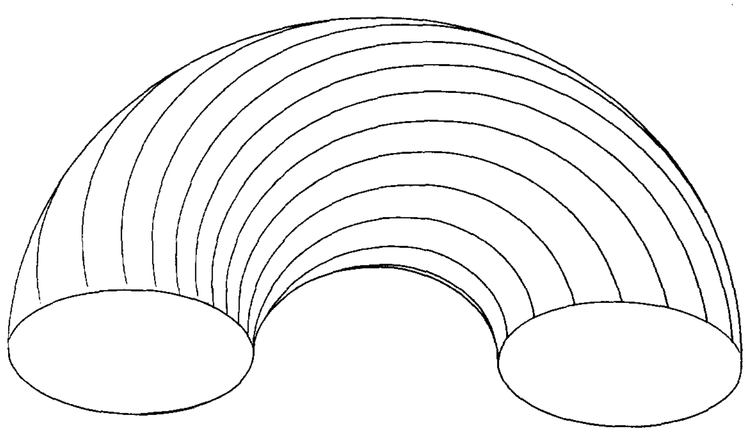

Figure 1. $R=1$; $a=0.5$; $b=0.5$; $\alpha=1$.

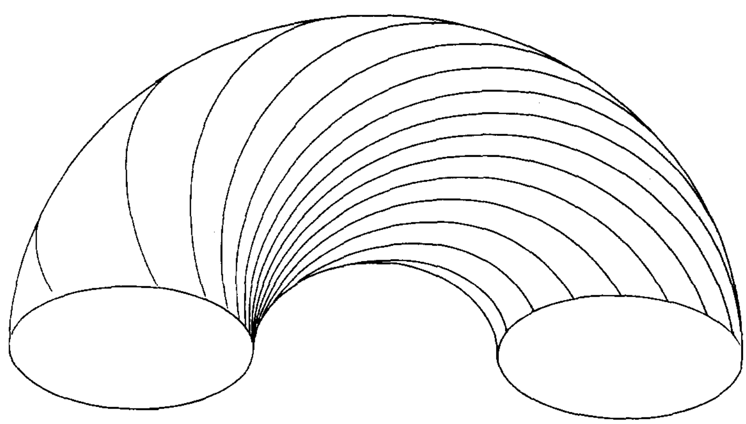

Figure 2. $R=1$; $a=0.5$; $b=0.5$; $\alpha=2$.

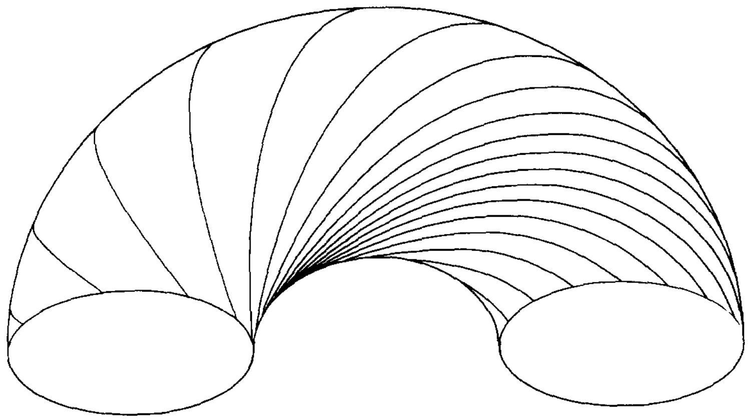

Figure 3. $R=1$; $a=0.5$; $b=0.5$; $\alpha=3$.

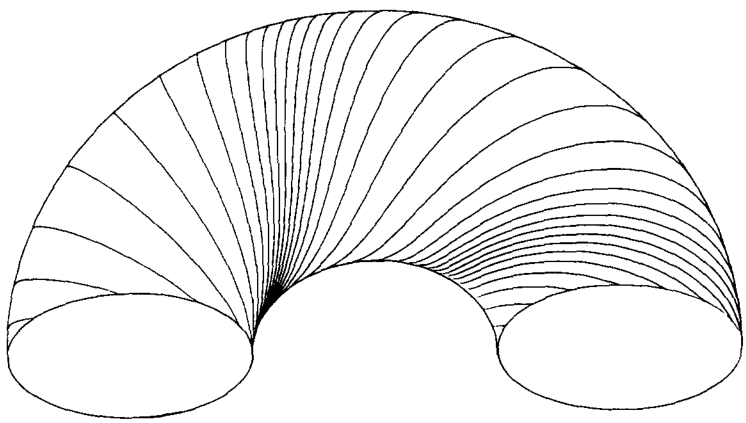

Figure 4. $R=1$; $a=0.5$; $b=0.5$; $\alpha=4$.

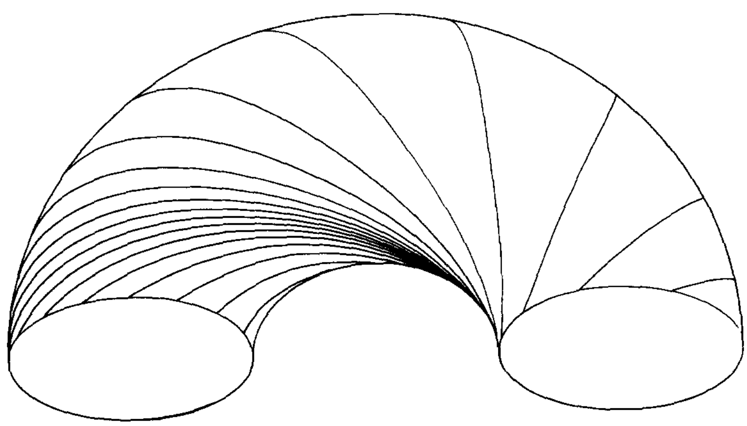

Figure 5. $R=1$; $a=0.5$; $b=0.5$; $\alpha=6$.

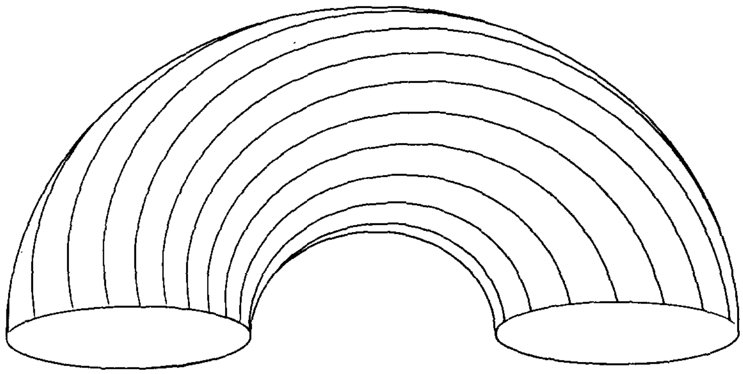

Figure 6. $R=1$; $a=0.5$; $b=0.25$; $\alpha=1$.

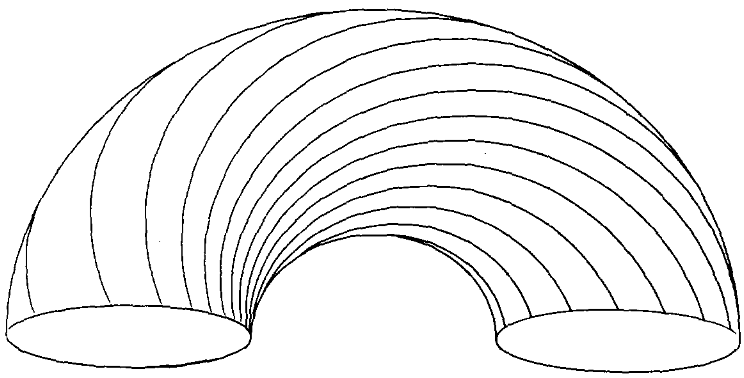

Figure 7. $R=1$; $a=0.5$; $b=0.25$; $\alpha=2$.

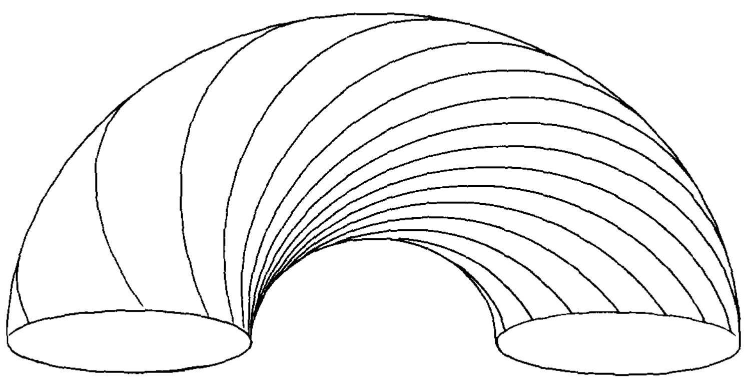

Figure 8. $R=1$; $a=0.5$; $b=0.25$; $\alpha=3$.

for approximate values δ_k for the solution $\tilde{\delta}(\sigma_k)$ with $\eta_k=\tilde{\eta}(\sigma_k)$. By the usual error analysis (see [1]) for Nyström's method for weakly singular integral equations of the second kind the behaviour of the quadrature error for (4.10) and (4.11) carries over to the error of the approximate solution to (4.1) obtained from (4.12). In particular, this means that doubling the number $2N$ of knots will double the number of correct digits in the approximate solution.

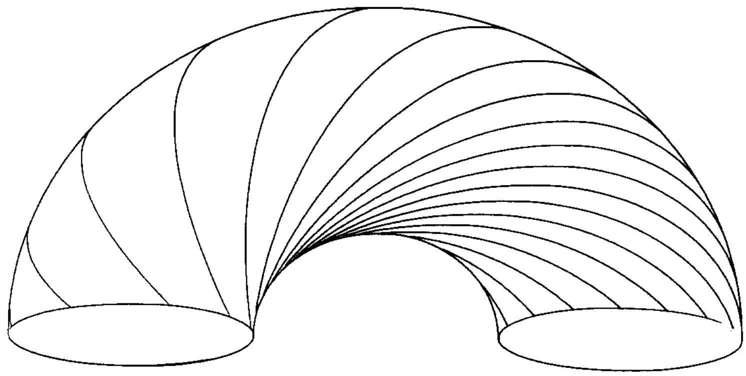

Figure 9. $R=1$; $a=0.5$; $b=0.25$; $\alpha=4$.

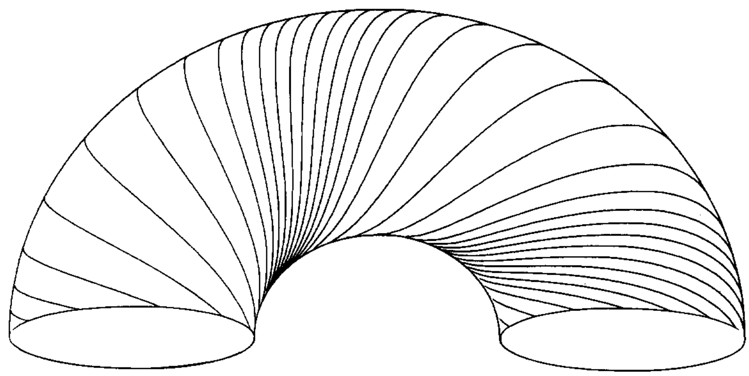

Figure 10. $R=1$; $a=0.5$; $b=0.25$; $\alpha=6$.

For the numerical example we choose an ellipse

$$r(\sigma)=R+a\cos\sigma,\quad z(\sigma)=b\sin\sigma$$

as cross-section. We illustrate the numerical results through Figures 1 to 10 showing the field lines on the boundary for $r=1$ and for $\alpha=1, 2, 3, 4$ and 6 in the case of a circle $a=0.5$ and $b=0.5$ and an ellipse $a=0.5$ and $b=0.25$.

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